

Homework 1

1. **Estimating logarithm function.** For $x \in [0, 1)$, we shall use the identity that

$$\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots.$$

- (a) **(5 points)** Prove that $\ln(1 - x) \leq -x - \frac{x^2}{2}$.

Solution.

(b) **(10 points)** For $x \in [0, 1/2]$, prove that

$$\ln(1-x) \geq -x - \frac{x^2}{2 \cdot 2^0} - \frac{x^2}{2 \cdot 2^1} - \frac{x^2}{2 \cdot 2^2} - \frac{x^2}{2 \cdot 2^3} - \dots = -x - x^2.$$

Solution.

2. **Tight Estimations.** Provide meaningful upper-bounds and lower-bounds for the following expressions.

(a) **(10 points)** $S_n = \sum_{i=1}^n \ln i$,

Solution.

(b) (10 points) $A_n = n!$

Solution.

(c) (10 points) $B_n = \binom{2n}{n} = \frac{(2n)!}{(n!)^2}$

Solution.

3. **Understanding Joint Distribution.** Recall that in the lectures we considered the joint distribution (\mathbb{T}, \mathbb{B}) over the sample space $\{4, 5, \dots, 10\} \times \{\mathbb{T}, \mathbb{F}\}$, where \mathbb{T} represents the time I wake up in the morning, and \mathbb{B} represents whether I have breakfast or not. The following table summarizes the joint probability distribution.

| t | b | $\mathbb{P}[\mathbb{T} = t, \mathbb{B} = b]$ |
|-----|--------------|--|
| 4 | \mathbb{T} | 0.01 |
| 4 | \mathbb{F} | 0.05 |
| 5 | \mathbb{T} | 0 |
| 5 | \mathbb{F} | 0.04 |
| 6 | \mathbb{T} | 0.1 |
| 6 | \mathbb{F} | 0.20 |
| 7 | \mathbb{T} | 0.25 |
| 7 | \mathbb{F} | 0.10 |
| 8 | \mathbb{T} | 0.10 |
| 8 | \mathbb{F} | 0.05 |
| 9 | \mathbb{T} | 0.03 |
| 9 | \mathbb{F} | 0.05 |
| 10 | \mathbb{T} | 0.01 |
| 10 | \mathbb{F} | 0.01 |

Calculate the following probabilities.

- (a) **(5 points)** Calculate the probability that I wake up at 8 a.m. or earlier, but do not have breakfast. That is, calculate $\mathbb{P}[\mathbb{T} \leq 8, \mathbb{B} = \mathbb{F}]$,

Solution.

- (b) **(5 points)** Calculate the probability that I wake up at 8 a.m. or earlier. That is, calculate $\mathbb{P}[\mathbb{T} \leq 8]$,

Solution.

- (c) **(5 points)** Calculate the probability that I skip breakfast conditioned on the fact that I woke up at 8 a.m. or earlier. That is, compute $\mathbb{P}[\mathbb{B} = \mathbb{F} \mid \mathbb{T} \leq 8]$.

Solution.

4. **Random Walk.** There is a frog sitting at the origin $(0, 0)$ in the first quadrant of a two-dimensional Cartesian plane. The frog first jumps uniformly at random along the X-axis to some point $(\mathbb{X}, 0)$, where $\mathbb{X} \in \{1, 2, 3, 4, 5, 6\}$. Then, it jumps uniformly at random along the Y-axis to some point (\mathbb{X}, \mathbb{Y}) , where $\mathbb{Y} \in \{1, 2, 3, 4, 5, 6\}$. So (\mathbb{X}, \mathbb{Y}) represents the final position of the frog after these two jumps. Note that \mathbb{X} and \mathbb{Y} are two independent random variables that are uniformly distributed over their respective sample spaces.

- (a) **(5 points)** What is the probability that the frog jumps more than 3 units along the Y-axis. That is, compute $\mathbb{P}[\mathbb{Y} > 3]$.

Solution.

- (b) **(10 points)** What is the probability that the final position of the frog is above the line $X + Y = 7$. That is compute $\mathbb{P}[\mathbb{X} + \mathbb{Y} > 7]$?

Solution.

- (c) **(10 points)** What is the probability that the frog has jumped 4 units along X-axis conditioned on the fact that its final position is above the line $X + Y = 7$? That is, compute $\mathbb{P}[X = 4 \mid X + Y > 7]$?

5. **Coin Tossing Word Problem.** We have three (independent) coins represented by random variables $\mathbb{C}_1, \mathbb{C}_2$, and \mathbb{C}_3 .

- (i) The first coin has $\mathbb{P}[\mathbb{C}_1 = H] = \frac{1}{4}, \mathbb{P}[\mathbb{C}_1 = T] = \frac{3}{4}$,
- (ii) The second coin has $\mathbb{P}[\mathbb{C}_2 = H] = \frac{3}{4}$ and $\mathbb{P}[\mathbb{C}_2 = T] = \frac{1}{4}$, and
- (iii) The third coin has $\mathbb{P}[\mathbb{C}_3 = H] = \frac{1}{4}$ and $\mathbb{P}[\mathbb{C}_3 = T] = \frac{3}{4}$.

Consider the following experiment.

- (A) Toss the first coin. Let the outcome of the first coin-toss be ω_1 .
- (B) If $\omega_1 = H$, then we toss the second coin twice. Otherwise, (i.e., if $\omega_1 = T$) toss the third coin twice. Let the two outcomes of this step be represented by ω_2 and ω_3 .
- (C) Output $(\omega_1, \omega_2, \omega_3)$.

Based on this experiment, compute the probabilities below.

- (a) **(5 points)** In the experiment mentioned above, what is the probability that a majority of the three outcomes $(\omega_1, \omega_2, \omega_3)$ are H (head)?

Solution.

- (b) **(5 points)** In the experiment mentioned above, what is the probability that a majority of the three outcomes are H , conditioned on the fact that the first outcome was T ?

Solution.

- (c) **(5 points)** In the experiment mentioned above, what is the probability that a majority of the three outcomes are different from the first outcome?

Solution.